Higher-rank representations for zero-spin field theories

1982 J. Phys. A: Math. Gen. 15 627

(http://iopscience.iop.org/0305-4470/15/2/029)

J. Phys. A: Math. Gen. 15 (1982) 627–635. Printed in Great Britain

Higher-rank representations for zero-spin field theories

W Cox
Department of Mathematics, University of Aston in Birmingham, Gosta Green, Birmingham B4 7ET, England

Received 10 June 1981, in final form 6 August 1981

Abstract. Field theories describing particles with zero spin, but utilising higher-rank Lorentz representations than usual, are considered. Of the three new first-order formulations given, two have consistent and causal minimal couplings.

1. Introduction

Recently, because of possible applications in supergravity, there has been some interest in theories of a given spin which involve higher-rank tensors or spinors than would be usual for that spin (Duff and Van Niewenhuizen 1980, Deser and Witten 1981, Siegel 1981, Deser *et al* 1980). Most of this work relates to gauge theories, in which the gauge invariance is used to cut out the unwanted higher spins—contrary to the usual usage in removing lower-spin states. However, the current interest reminds us that little work has been done on massive theories of this type. Almost invariably in massive field theories one tends to take the minimum-rank tensor/spinor field which contains the required physical spin. Deser and co-workers (Deser and Witten 1981, Deser *et al* 1980) have considered the use of an antisymmetric second-rank tensor to describe a spin-zero field and found that in the massive theory it appears to be impossible to covariantly ensure non-trivial dynamics. The purpose of this paper is to show how a massive spin-zero field may be described by a Lagrangian using higher-rank tensors, either by introducing extra auxiliary tensor fields or by doubling up on say the vector auxiliary field. The behaviour of the theories on minimal coupling to the electromagnetic field is discussed. When higher-rank representations are present in a unique mass–spin theory one usually encounters various interaction difficulties such as loss of constraints or acausality (Velo and Zwanzinger 1969a, b, Mathews *et al* 1980), and we study this problem for the new theories obtained.

2. Lagrangians using higher-rank representations

For definiteness, we consider Lagrangians and field equations which are at most first order in derivatives. Any higher-order system can be reduced to first order by introducing auxiliary fields and constraints. In this process higher-rank tensors are automatically generated—one vector index for each derivative order we reduce by. Of course, this is not in general what we mean by using higher-rank tensors to represent lower spin. To be specific about what constitutes such a theory we consider the Lorentz

0305-4470/82/020627+09$02.00

irreps which are carried by the field variable ψ in the typical first-order system

$$(L_\mu \partial^\mu + i\chi)\psi = 0 \tag{2.1}$$

where L_μ are square matrices and χ is a non-zero scalar matrix.

The theory of such systems is well known (Gel'fand *et al* 1963). All the physical content of the free field is contained in the structure of the matrix L_0. Choosing a representation for ψ in which all components corresponding to the same spin are collected together, L_0 assumes a block diagonal form, with the 's blocks' on the diagonal determining the possible mass–spin states of the field. A non-zero eigenvalue λ of an s block corresponds to a spin-s state of the field with mass proportional to χ/λ. The zero eigenvalues of the s blocks correspond to constraints.

If we require a theory of a field with unique spin $s = j$, the j block must have a non-zero eigenvalue, while all other blocks must be nilpotent. For a unique mass particle–antiparticle spin-j pair the j block must have eigenvalues ± 1, and the remaining s blocks must be nilpotent. Given the (reducible) Lorentz representation carried by ψ, and the mass–spin states required, we can write down the most general possible L_0 consistent with covariance, parity invariance and Lagrangian origin and then study the s blocks to see how, if possible, the required mass–spin spectra may be obtained. This s-block analysis provides a systematic way of determining which field representations and which equations can yield theories of fields with specified mass and spin. Graphical methods have been developed to assist in this analysis (Cox 1974a, b, c, 1978, 1981, 1982).

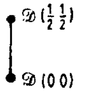
Normally, if one wants say a unique mass–spin theory of spin j one chooses the representation carried by ψ to contain at most up to spin j (except in the case of spin zero, where we take up to spin one), and in such a way that the j block has eigenvalues ± 1 and zero, and all lower-spin blocks are nilpotent. This would correspond in the usual field theory approach to taking tensors of just sufficient rank to accomodate the required physical spin. For the exceptional case of the first-order formulation of a spin-zero field theory this requires taking at most a vector and a scalar field in the Lagrangian. In this case the spin-one block is zero.

In the theories we now have in mind the $j+1$ block, and possibly higher, would be allowed to be non-zero but nilpotent, as well as those below spin j. This is not necessarily equivalent to using higher-derivative Lagrangians, because, as Deser *et al* (1980) observe, the physical content of a theory is not always left unchanged by field substitutions involving derivatives. There is no compelling reason why such theories using higher-rank representations than necessary should show improved behaviour, say with respect to interactions, but it is interesting that so far the only theory with a consistent, causal minimal coupling to the electromagnetic field with L_0 non-diagonalisable is precisely such a theory, albeit only spin one (Shamaly and Capri 1973).

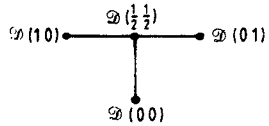
The sort of s-block analysis described above tells us what Lorentz irreps are involved in ψ and how these are linked by derivatives in equation (2.1) to yield the desired mass–spin spectra. For interaction analysis however, it is more convenient to have the equations in tensor or tensor–spinor form. This can be achieved by introducing appropriate tensors/spinors to represent the various Lorentz irreps and linking these by derivatives according to the structure of the graph representing L_0 (Cox 1982). In the present analysis the graphs depicted may be simply interpreted in the following way. The graph as a whole represents the full set of field equations. The vertices denote irreps of the proper Lorentz group, represented in turn by some appropriate irreducible tensor field. The edge between two vertices represents differentiation. If a vertex A is connected by an edge to vertex B then in the field equation for B the contribution of field A is got by differentiating the field A and multiplying by an arbitrary complex parameter. These parameters are subject to certain relations by the requirements of space reflection covariance, hermiticity etc. With the use of certain conventions the s blocks of a theory correspond to particular subgraphs of the graph, as described in Cox (1974a, b, c, 1978). Information on the possible mass–spin spectra of a theory may be gleaned by visual inspection of these subgraphs. In particular the very form of an s-block graph may preclude the nilpotency of that s block. While such graphical techniques are useful in deciding what structure of equations for a given combination of irreps will yield the required mass–spin spectra, they are not necessary for the subsequent analysis of these equations, which is the object of this paper. Further details of the graphical methods can be found in the references.

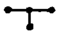
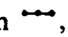
3. Alternative forms for massive spin-zero theories

In graphical notation (Cox 1974a, b, c) the usual first-order formulation of spin-zero theory of Duffin and Kemmer (Duffin 1938, Kemmer 1939) corresponds to the graph

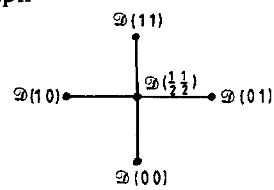


The 0 block is 2×2 and chosen to have eigenvalues ± 1, while the 1 block is zero. The theory of Deser and Witten (1981) which uses just a vector A_μ and antisymmetric tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ corresponds to the graph



so far as the kinematic part of the Lagrangian is concerned, but except in the case $\alpha = 0$ the mass terms in the Lagrangian cannot be put into the form (2.1), but correspond to the interesting but little studied case where χ is a singular matrix. In the case $\alpha \neq 0$, the only possibilities for a spin-zero theory lead in fact to trivial dynamics. In the $\alpha = 0$ case the only possible Lagrangian of form (2.1) is parity invariant, and it is then easy to see from the s blocks that the graph  cannot describe a single spin-zero state. The 0 block can be made massive, but the 1 block  cannot be made nilpotent. In fact the graph  describes a theory with massive spin-zero and spin-one states. These contribute to the energy with different signs and since neither s block can be made nilpotent one or the other must be made zero, resulting in either a graph , the spin-zero theory, or a graph , the spin-one theory.

To obtain a modification of the simple massive spin-zero theory, we need to introduce other higher-rank tensors, at the same time ensuring that we still have a nilpotent 1 block. One way to do this is by introducing the representation $\mathscr{D}(11)$ and using the graph



The s-block analysis confirms that such a theory can describe a unique spin-zero state, and that up to rescaling of the fields the theory is unique.

To obtain this theory in tensor form we represent $\mathscr{D}(11)$ by a symmetric traceless tensor $S_{\mu\nu}$ and $\mathscr{D}(101)\equiv\mathscr{D}(10)\oplus\mathscr{D}(01)$ by an antisymmetric tensor $A_{\mu\nu}$. A_μ and ϕ represent $\mathscr{D}(\frac{1}{2}\frac{1}{2})$ and $\mathscr{D}(00)$ respectively. The above graph then represents the equations

$$S_{\mu\nu}=a_1\{\partial_\mu A_\nu\}_{ST}=\tfrac{1}{2}a_1(\partial_\mu A_\nu+\partial_\nu A_\mu-\tfrac{1}{2}g_{\mu\nu}\partial\cdot A) \tag{3.1}$$

$$A_\mu=\tilde{a}_1\partial^\nu S_{\mu\nu}+\tilde{a}_2\partial^\nu A_{\mu\nu}+a_3\partial_\mu\phi \tag{3.2}$$

$$A_{\mu\nu}=a_2(\partial_\mu A_\nu-\partial_\nu A_\mu) \tag{3.3}$$

$$\phi=\tilde{a}_3\partial\cdot A \tag{3.4}$$

where a_i, $\tilde{a}_i$ are independent constants, and if parity invariance and Lagrangian origin are assumed $\tilde{a}_i=\pm a_i$ (Cox 1974a, b, c) although we shall not yet make either of these assumptions.

Eliminating $S_{\mu\nu}$ and $A_{\mu\nu}$ from (3.2)

$$A_\mu=(\tfrac{1}{2}x_1-x_2)\partial^2 A_\mu+(\tfrac{1}{4}x_1+x_2+x_3)\partial_\mu(\phi/a_3)$$

where $x_i=\tilde{a}_i a_i$. The other apparent spin-one modes in the theory, $\partial^\nu S_{\mu\nu}$ and $\partial^\nu A_{\mu\nu}$ depend only on A_μ as do $A_{\mu\nu}$ and $S_{\mu\nu}$. All physical spin-one modes can therefore be eliminated by ensuring that A_μ does not propagate, i.e. by taking

$$x_1=2x_2 \tag{3.5}$$

then

$$A_\mu=a_3^{-1}(\tfrac{3}{2}x_2+x_3)\partial_\mu\phi. \tag{3.6}$$

So A_μ really conceals a physical spin-zero field. Further, from (3.4)

$$\partial\cdot A=a_3^{-1}\phi=a_3^{-1}(\tfrac{3}{2}x_2+x_3)\partial^2\phi$$

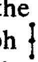
so ϕ propagates according to

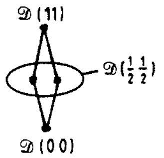
$$[\partial^2-(\tfrac{3}{2}x_2+x_3)^{-1}]\phi=0. \tag{3.7}$$

So we simply have to take

$$\tfrac{3}{2}x_2+x_3<0 \tag{3.8}$$

for a propagating physical spin-zero massive state.

Thus, the system (3.1)–(3.4), with (3.5), (3.8) provides us with a massive free field theory of spin zero using higher than necessary Lorentz representations. However, we shall see in the next section that in the case of minimal coupling to an external electromagnetic field this theory is inconsistent. The problem lies in the way the contributions of A_μ to $S_{\mu\nu}$ and $A_{\mu\nu}$ combine to eliminate the spin-one state. However, if we try to eliminate either of these then the 1 block cannot be made nilpotent. For example, eliminating $A_{\mu\nu}$ gives the graph , the 1 block of which, , cannot be made nilpotent. However, by doubling up the $\mathscr{D}(\frac{1}{2}\frac{1}{2})$ representation we can achieve the required mass–spin spectra with the graph



The s-block analysis confirms that the parameters in this theory can be chosen to yield a nilpotent 1 block and a 0 block with two non-zero eigenvalues ± 1. In tensor form it corresponds to the equations

$$S_{\mu\nu} = b_1\{\partial_\mu A_\nu\}_{ST} + b_2\{\partial_\mu B_\nu\}_{ST} \tag{3.9}$$

$$A_\mu = \tilde{b}_1 \partial^\nu S_{\mu\nu} + b_3 \partial_\mu \phi \tag{3.10}$$

$$B_\mu = \tilde{b}_2 \partial^\nu S_{\mu\nu} + b_4 \partial_\mu \phi \tag{3.11}$$

$$\phi = \tilde{b}_3 \partial \cdot A + \tilde{b}_4 \partial \cdot B. \tag{3.12}$$

Eliminating $S_{\mu\nu}$ and combining (3.10), (3.11) yields ($y_i = \tilde{b}_i b_i$)

$$b_1 A_\mu + b_2 B_\mu = \tfrac{1}{2}(y_1 + y_2)\partial^2(b_1 A_\mu + b_2 B_\mu)$$
$$+ \tfrac{1}{4}(y_1 + y_2)\partial_\mu[\partial \cdot (b_1 A + b_2 B)] + (b_1 b_3 + b_2 b_4)\partial_\mu \phi$$

and to avoid a propagating spin-one mode we must take

$$y_1 + y_2 = 0. \tag{3.13}$$

Then

$$b_1 A_\mu + b_2 B_\mu = (b_1 b_3 + b_2 b_4)\partial_\mu \phi \tag{3.14}$$

and this in (3.10), (3.11) expresses both A_μ, B_μ (and hence $S_{\mu\nu}$) entirely in terms of ϕ, the spin-zero field.

For the spin-zero modes in the theory, (3.9)–(3.12) yield

$$\partial^\mu \partial^\nu S_{\mu\nu} = \tfrac{3}{4} b_1 \partial^2 (\partial \cdot A) + \tfrac{3}{4} b_2 \partial^2 (\partial \cdot B) \tag{3.15}$$

$$\partial \cdot A = \tilde{b}_1 \partial^\mu \partial^\nu S_{\mu\nu} + b_3 \partial^2 \phi \tag{3.16}$$

$$\partial \cdot B = \tilde{b}_2 \partial^\mu \partial^\nu S_{\mu\nu} + b_4 \partial^2 \phi \tag{3.17}$$

$$\phi = \tilde{b}_3 \partial \cdot A + \tilde{b}_4 \partial \cdot B. \tag{3.18}$$

Eliminating $\partial^\mu \partial^\nu S_{\mu\nu}$, $\partial \cdot A$ and $\partial \cdot B$ yields

$$\phi = \tfrac{3}{4}(\tilde{b}_1 \tilde{b}_3 + \tilde{b}_2 \tilde{b}_4)(b_1 b_3 + b_2 b_4)\partial^4 \phi + (y_3 + y_4)\partial^2 \phi \tag{3.19}$$

on using (3.13). For a unique massive spin-zero mode we must therefore also insist on

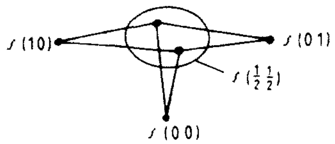
$$b_1 b_3 + b_2 b_4 = 0 \qquad \text{or} \qquad \tilde{b}_1 \tilde{b}_3 + \tilde{b}_2 \tilde{b}_4 = 0 \tag{3.20}$$

while

$$y_3 + y_4 < 0. \tag{3.21}$$

So the system (3.9)–(3.12), with the $\tilde{b}_i$, b_i solutions of (3.13), (3.20), (3.21), which they clearly can be, will give a theory of a massive spin-zero field. Note that if we take $b_1 b_3 + b_2 b_4 = 0$, then (3.14) reduces to $b_1 A_\mu + b_2 B_\mu = 0$, which in (3.9) yields $S_{\mu\nu} = 0$ as a consequence of the field equations.

By doubling up the $\mathscr{D}(\frac{1}{2}\frac{1}{2})$ representation we can also obtain a massive spin-zero theory using the antisymmetric tensor via the graph



Again, a simple s-block analysis allows a nilpotent 1 block and massive 0 block. In tensor form the equations are

$$A_{\mu\nu} = c_1(\partial_\mu A_\nu - \partial_\nu A_\mu) + c_2(\partial_\mu B_\nu - \partial_\nu B_\mu) \tag{3.22}$$

$$A_\mu = \tilde{c}_1 \partial^\nu A_{\mu\nu} + c_4 \partial_\mu \phi \tag{3.23}$$

$$B_\mu = \tilde{c}_2 \partial^\nu A_{\mu\nu} + c_3 \partial_\mu \phi \tag{3.24}$$

$$\phi = \tilde{c}_4 \partial^\mu A_\mu + \tilde{c}_3 \partial^\mu B_\mu \tag{3.25}$$

c_i, $\tilde{c}_i$ again some constants, with no *a priori* relation yet assumed between c_i and $\tilde{c}_i$.

Eliminating $A_{\mu\nu}$ and combining (3.23) and (3.24) yields the result ($z_i = \tilde{c}_i c_i$)

$$c_1 A_\mu + c_2 B_\mu = (z_1 + z_2)\partial_\mu[\partial \cdot (c_1 A + c_2 B)]$$
$$\qquad - (z_1 + z_2)\partial^2(c_1 A_\mu + c_2 B_\mu) + (c_1 c_4 + c_2 c_3)\partial_\mu \phi.$$

To avoid a propagating spin-one mode we must choose

$$z_1 + z_2 = 0 \tag{3.26}$$

and then

$$c_1 A_\mu + c_2 B_\mu = (c_1 c_4 + c_2 c_3)\partial_\mu \phi \tag{3.27}$$

and (3.22) reduces to

$$A_{\mu\nu} = 0$$

and (3.23), (3.24) to

$$A_\mu = c_4 \partial_\mu \phi \qquad B_\mu = c_3 \partial_\mu \phi$$

again, as consequences of the field equations. For the spin-zero modes we have

$$\partial^\mu \partial^\nu A_{\mu\nu} = 0 \qquad \partial \cdot A = c_4 \partial^2 \phi \qquad \partial \cdot B = c_3 \partial^2 \phi$$

and

$$\phi = (z_3 + z_4)\partial^2 \phi.$$

So we have a free field theory using the second-rank antisymmetric tensor $A_{\mu\nu}$ which describes a propagating massive spin-zero field. To pay for this we have had to introduce a second vector field in the constraints. However, we will find, as in the case of the graph , that there is a pay-off in interaction consistency.

4. Minimal coupling

The simplest spin-zero first-order theory—the five-dimensional Duffin–Kemmer theory—suffers no problems such as loss of constraints or acausality under minimal coupling. This is essentially because the L_0 matrix is in this case diagonalisable (Amar and Dozzio 1972, Velo and Zwanzinger 1971). It is clear that for our theories using higher-rank representations the L_0 matrix *cannot* be diagonalisable, since the 1 block must be nilpotent but non-zero, leading to a nilpotency index greater than one (Cox 1981). Usually such theories suffer various types of problems such as lack of constraints or acausality on minimal coupling (Velo and Zwanzinger 1969a, b, 1971, Mathews *et al* 1980), and we therefore need to examine this question for our theories. At present there are only a few theories known with non-diagonalisable L_0 which have consistent and causal minimal couplings (Shamaly and Capri 1973, Khalil 1977). In this section we add two more such theories to the list.

For the  theory the equations (3.1)–(3.4) become on minimal coupling, $\partial_\mu \to \pi_\mu = \partial_\mu - ie\phi_\mu$, ϕ_μ the electromagnetic potential:

$$S_{\mu\nu} = \tfrac{1}{2}a_1(\pi_\mu A_\nu + \pi_\nu A_\mu - \tfrac{1}{2}g_{\mu\nu}\pi \cdot A) \tag{4.1}$$

$$A_\mu = \tilde{a}_1 \pi^\nu S_{\mu\nu} + \tilde{a}_2 \pi^\nu A_{\mu\nu} + a_3 \pi_\mu \phi \tag{4.2}$$

$$A_{\mu\nu} = a_2(\pi_\mu A_\nu - \pi_\nu A_\mu) \tag{4.3}$$

$$\phi = \tilde{a}_3 \pi \cdot A. \tag{4.4}$$

If we now try to eliminate $S_{\mu\nu}$ and $A_{\mu\nu}$ as before, we obtain, using (3.5)

$$A_\mu = a_3^{-1}(\tfrac{3}{2}x_2 - x_3)\pi_\mu \phi + 2x_2 F^\nu{}_\mu A_\nu$$

where

$$F^\nu{}_\mu = [\pi^\nu, \pi_\mu].$$

In the free field case this corresponds to the constraint (3.6), expressing A_μ entirely in terms of the spin-zero field. However, in a non-zero electromagnetic field we obtain

$$(\delta^\nu{}_\mu - 2x_2 F^\nu{}_\mu)A_\nu = a_3^{-1}(\tfrac{3}{2}x_2 + x_3)\pi_\mu \phi. \tag{4.5}$$

Now if x_2 is real, as it is for systems derivable from a real non-degenerate parity invariant Lagrangian, then there exist Lorentz frames in which $\delta^\nu{}_\mu - 2x_2 F^\nu{}_\mu$ is singular, and so we cannot covariantly solve the constraint (4.5) for A_μ in the presence of a non-zero field. Thus, in this case the constraint structure is changed by the dynamics and the interactions of the theory are inconsistent. The difficulty is that the free field requirements on x_1, x_2 (i.e. (3.5)) preclude the elimination of the troublesome $F^\nu{}_\mu A_\nu$ term in the interaction case. This difficulty does not arise in the other two theories considered in § 3.

Thus, consider the  theory. Equations (3.9)–(3.12) become

$$S_{\mu\nu} = \tfrac{1}{2}b_1\{\pi_\mu A_\nu\}_{ST} + \tfrac{1}{2}b_2\{\pi_\mu B_\nu\}_{ST} \tag{4.6}$$

$$A_\mu = \tilde{b}_1 \pi^\nu S_{\mu\nu} + b_3 \pi_\mu \phi \tag{4.7}$$

$$B_\mu = \tilde{b}_2 \pi^\nu S_{\mu\nu} + b_4 \pi_\mu \phi \tag{4.8}$$

$$\phi = \tilde{b}_3 \pi \cdot A + \tilde{b}_4 \pi \cdot B. \tag{4.9}$$

With (3.13) we still obtain the direct generalisation of (3.14)

$$b_1 A_\mu + b_2 B_\mu = (b_1 b_3 + b_2 b_4)\pi_\mu \phi. \tag{4.10}$$

Back substituting into (4.6)–(4.8) gives $S_{\mu\nu}$, A_μ, B_μ as functions of ϕ only, as before.

Also, for the spin-zero modes we have

$$\pi^\mu \pi^\nu S_{\mu\nu} = b_1 \pi^\mu \pi^\nu \{\pi_\mu A_\nu\}_{ST} + b_2 \pi^\mu \pi^\nu \{\pi_\mu B_\nu\}_{ST} \tag{4.11}$$

$$\pi \cdot A = \tilde{b}_1 \pi^\mu \pi^\nu S_{\mu\nu} + b_3 \pi^2 \phi \tag{4.12}$$

$$\pi \cdot B = \tilde{b}_2 \pi^\mu \pi^\nu S_{\mu\nu} + b_4 \pi^2 \phi \tag{4.13}$$

$$\phi = \tilde{b}_3 \pi \cdot A + \tilde{b}_4 \pi \cdot B. \tag{4.14}$$

Substituting (4.12), (4.13) into (4.14) gives

$$\phi = (\tilde{b}_1 \tilde{b}_3 + \tilde{b}_2 \tilde{b}_4) \pi^\mu \pi^\nu S_{\mu\nu} + (y_3 + y_4) \pi^2 \phi. \tag{4.15}$$

On the other hand, substituting (4.7), (4.8) in (4.11) we get, using (3.13)

$$\pi^\mu \pi^\nu S_{\mu\nu} = (b_1 b_3 + b_2 b_4) \pi^\mu \pi^\nu \{\pi_\mu \pi_\nu \phi\}_{ST}. \tag{4.16}$$

Since from (3.20), either $b_1 b_3 + b_2 b_4$ or $\tilde{b}_1 \tilde{b}_3 + \tilde{b}_2 \tilde{b}_4$ must be zero (4.15) and (4.16) must reduce to

$$\phi = (y_3 + y_4) \pi^2 \phi \tag{4.17}$$

so that ϕ propagates causally and consistently.

Turning now to the third theory equations (3.22)–(3.25) become

$$A_{\mu\nu} = c_1(\pi_\mu A_\nu - \pi_\nu A_\mu) + c_2(\pi_\mu B_\nu - \pi_\nu B_\mu) \tag{4.18}$$

$$A_\mu = \tilde{c}_1 \pi^\nu A_{\mu\nu} + c_4 \pi_\mu \phi \tag{4.19}$$

$$B_\mu = \tilde{c}_2 \pi^\nu A_{\mu\nu} + c_3 \pi_\mu \phi \tag{4.20}$$

$$\phi = \tilde{c}_4 \pi \cdot A + \tilde{c}_3 \pi \cdot B. \tag{4.21}$$

Just as in the free case, with $z_1 + z_2 = 0$ we obtain

$$c_1 A_\mu + c_2 B_\mu = (c_1 c_4 + c_2 c_3) \pi_\mu \phi \tag{4.22}$$

whence A_μ, B_μ, $A_{\mu\nu}$ are entirely expressible in terms of ϕ, and $A_{\mu\nu}$ vanishes in the free field limit, consistently with the constraint (3.28).

For the spin-zero modes we have

$$\pi \cdot A = \tfrac{1}{2} \tilde{c}_1 F^{\mu\nu} A_{\mu\nu} + c_4 \pi^2 \phi$$

$$\pi \cdot B = \tfrac{1}{2} \tilde{c}_2 F^{\mu\nu} A_{\mu\nu} + c_3 \pi^2 \phi$$

where $F^{\mu\nu} = [\pi^\mu, \pi^\nu]$, so

$$\phi = \tfrac{1}{2}(\tilde{c}_1 \tilde{c}_4 + \tilde{c}_2 \tilde{c}_3) F^{\mu\nu} A_{\mu\nu} + (z_3 + z_4) \pi^2 \phi.$$

Also, by substituting A_μ, B_μ into $A_{\mu\nu}$ and using $z_1 + z_2 = 0$ we obtain

$$A_{\mu\nu} = (c_1 c_4 + c_2 c_3) F_{\mu\nu} \phi$$

so

$$\pi^\mu \pi^\nu A_{\mu\nu} = \tfrac{1}{2} F^{\mu\nu} A_{\mu\nu} = \tfrac{1}{2}(c_1 c_4 + c_2 c_3) F^2 \phi.$$

Thus the equation for ϕ becomes

$$\phi = \tfrac{1}{2}(c_1 c_4 + c_2 c_3)(\tilde{c}_1 \tilde{c}_4 + \tilde{c}_2 \tilde{c}_3) F^2 \phi + (z_3 + z_4) \pi^2 \phi. \tag{4.23}$$

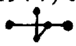
In this case we have a choice of theories depending on whether or not we assume

$$c_1 c_4 + c_2 c_3 = 0 \qquad \text{or} \qquad \tilde{c}_1 \tilde{c}_4 + \tilde{c}_2 \tilde{c}_3 = 0 \tag{4.24}$$

neither of which are required by the free field theory. If we do not assume either of these, then the minimally coupled first-order theory is equivalent to a spin-zero theory with non-minimal coupling, although ϕ still propagates consistently and causally.

5. Conclusion

We have obtained a number of alternative first-order spin-zero massive field theories involving higher representations of the Lorentz group than usual. One of these, that using an antisymmetric tensor, two vector fields and one scalar, appears to satisfy the requirements of Deser and Witten (1981), and furthermore is consistent under minimal coupling to the electromagnetic field.

The theories illustrate a number of interesting points relating to the high-spin interaction problem, even though they only describe zero spin. First of all, as the  theory shows, low spin is no guarantee of consistent minimal coupling—if higher-rank representations are present and the constraints are sufficiently awkward then problems can arise even for spin zero. The other two theories on the other hand illustrate again (Shamaly and Capri 1973, Khalil 1977) the fact that theories with non-diagonalisable L_0 need not be inconsistent. The  theory is particularly interesting because by inequivalent choices of the parameters to satisfy free field requirements we can obtain theories behaving differently on minimal coupling.

In these examples the use of repeated irreps provided a by-pass for the interaction difficulties, and perhaps similar possibilities may exist for higher spin.

References

Amar V and Dozzio U 1972 *Lett. Nuovo Cimento* **5** 355

Cox W 1974a *J. Phys. A: Math., Nucl. Gen.* **7** 1

——1974b *J. Phys. A: Math., Nucl. Gen.* **7** 66

——1974c *J. Phys. A: Math., Nucl. Gen.* **7** 2249

——1978 *J. Phys. A: Math. Gen.* **11** 1167

——1981 *J. Phys. A: Math. Gen.* **14** 2459

——1982 *J. Phys. A: Math. Gen.* **15** 253

Deser S, Townsend P K and Siegel W 1980 *CERN preprint* TH2998

Deser S and Witten E 1981 *Nucl. Phys.* **178B** 491

Duff M J and Van Nieuwenhuizen P 1980 *Phys. Lett.* **94B** 179

Duffin R J 1938 *Phys. Rev.* **54** 1114

Gel'fand I M, Minlos R A and Shapiro Z Ya 1963 *Representations of the Rotation and Lorentz Groups and their Applications* (Oxford: Pergamon)

Kemmer N 1939 *Proc. R. Soc.* A **173** 91

Khalil M A K 1977 *Phys. Rev.* D **15** 1532

Mathews P M, Govindarajan T R, Seetharaman M and Prabhakarran J 1980 *J. Math. Phys.* **21** 1495

Shamaly A and Capri A Z 1973 *Can. J. Phys.* **51** 1467

Siegel W 1981 *Caltech preprint* CALT-68-803

Velo G and Zwanzinger D 1969a *Phys. Rev.* **186** 1337

——1969b *Phys. Rev.* **188** 2218

——1971 *Proc. Coral Gables. Conf. on Fundamental Interactions* vol 4, ed M Dal Cin, G J Iverson and A Perlmutter (London: Gordon and Breach)